

D 132038

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Name.....

Reg. No.....

**THIRD SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY)
EXAMINATION, NOVEMBER 2025**

(CBCSS)

Mathematics

MTH 3C 11—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admission onwards)

Time : Three Hours

Maximum : 30 Weightage

Part A (Short Answer Type Questions)*Answer all questions.**Each question carries a weightage 1.*

1. Define vector space.
2. Define linear transformation between two vectors spaces. Illustrate with an example.
3. Define determinant as a real function on the set of all ordered n -tuples of vectors in $\mathbb{R}^n$.
4. Define parametrized curve in $\mathbb{R}^n$. Give a parametrization of the parabola $x^2 + y^2 = 1$.
5. Compute the curvature of the curve

$$\gamma(t) = (t, \cosh t).$$

6. Describe briefly : Stereographic projection from S^2 to the plane.
7. Let $\sigma(u, v)$ be a surface patch with standard unit normal $N(u, v)$. Then prove that $N_u \cdot \sigma_v = -M$.
8. Describe briefly : Gaussian curvature and mean curvature.

(8 × 1 = 8 weightage)

Turn over

Part B (Paragraph Type Questions)

Answer any **two** questions from each module.

Each question carries a weightage 2.

MODULE I

9. Prove that $\dim \mathbb{R}^n = n$.
10. Suppose f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E , and there is a real number M such that

$$\|f'(x)\| \leq M$$

for every $x \in E$. Then prove that

$$|f(b) - f(a)| \leq M |b - a|$$

for all $a \in E$ and $b \in E$.

11. If I is the identity operator on $\mathbb{R}^n$, then prove that

$$\det[I] = \det(e_1, \dots, e_n) = 1.$$

MODULE II

12. Find the tangent vector of the limaçon

$$\gamma(t) = ((1 + 2 \cos t) \cos t, (1 + 2 \cos t) \sin t).$$

What is the tangent vector of this curve at the origin?

13. Let $\gamma: (\alpha, \beta) \rightarrow \mathbb{R}^2$ be a unit-speed curve, let $s_0 \in (\alpha, \beta)$ and let φ_0 be such that

$$\dot{\gamma}(s_0) = (\cos \varphi_0, \sin \varphi_0).$$

Then prove that there is a unique smooth function $\varphi: (\alpha, \beta) \rightarrow \mathbb{R}$ such that $\varphi(s_0) = \varphi_0$ and that the equation

$$\dot{\gamma}(s) = (\cos \varphi(s), \sin \varphi(s)).$$

holds for all $s \in (\alpha, \beta)$.

14. Let $\sigma : U \rightarrow \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$, and let (u, v) be co-ordinates in U . Then prove that the tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v .

MODULE III

15. Prove that the area of a surface patch is unchanged by reparametrization.
16. If σ is a surface patch of S and $\gamma(t) = \sigma(u(t), v(t))$ is a curve in σ , then find κ_n .
17. Prove that the principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point. Also prove that the principal vectors are the tangent vectors of the curves giving these maximum and minimum values.

(6 × 2 = 12 weightage)

Part C (Essay Type Questions)

Answer any **two** questions.

Each question carries a weightage 5.

18. (a) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$, and c is a scalar, then prove that

$$\|A + B\| \leq \|A\| + \|B\|$$

and

$$\|cA\| = |c| \|A\|.$$

- (b) Suppose f maps on an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Then prove that $f \in C'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.
19. (a) Prove that a parametrized curve has a unit-speed reparametrization if and only if it is regular.
- (b) If $f : S_1 \rightarrow S_2$ is a diffeomorphism, then prove that for all $p \in S_1$ the linear map $D_p f : T_p S_1 \rightarrow T_{f(p)} S_2$ is invertible.

Turn over

20. Prove that a local diffeomorphism $f : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is conformal if and only if there is a function $\lambda : \mathcal{S}_1 \rightarrow \mathbb{R}$ such that

$$f^* \langle v, w \rangle_p = \lambda(p) \langle v, w \rangle_p$$

for all $p \in \mathcal{S}_1$ and $v, w \in T_p \mathcal{S}_1$.

21. (a) Prove that a local diffeomorphism $f : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is a local isometry if and only if, for any surface patch σ_1 of $\mathcal{S}_1$, the patches σ_1 and $f \circ \sigma_1$ of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, have the same first fundamental form.
- (b) Let S be a (connected) surface of which every point is an umbilic. Then, prove that S is an open subset of a plane or a sphere.

(2 × 5 = 10 weightage)