D 132039 **(Pages : 5)** **Name.......................................**

Reg. No...................................

THIRD SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, NOVEMBER 2025

(CBCSS)

Mathematics

MTH 3C 12—COMPLEX ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A (Short Answer Type Questions)

Answer **all** *questions.*
Each question carries a weightage 1.

1. If $f : G \to \mathbb{C}$ is differentiable at a point a in G then prove that f is continuous at a.

2. If S is a Möbius transformation then prove that S is the composition of translations, dilations, and the inversion.

3. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$.

4. Let γ be a rectifiable curve in $\mathbb{C}$ and suppose that F_n and F are continuous functions on $\{\gamma\}$. If

$$F = u - \lim F_n$$

on $\{\gamma\}$ then prove that

$$\int_{\gamma} \lim F_n = \lim \int_{\gamma} F_n.$$

5. If $\gamma : [0, 1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$ then prove that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}$$

is an integer.

Turn over

6. Evaluate $\int_{\gamma} \frac{2z}{z^2 + z + 1} dz$ where γ is the circle $|z| = 2$.

7. Prove that a function $f : [a, b] \to \mathbb{R}$ is convex if and only if the set

$$A = \{(x, y) : a \le x \le b \text{ and } f(x) \le y\}$$

is convex.

8. State Schwarz's Lemma.

(8 × 1 = 8 weightage)

Part B (Paragraph Type Questions)

Answer any **two** *questions from each module.*
Each question carries a weightage 2.

Module I

9. Let $\sum a_n (z-a)^n$ and $\sum b_n (z-a)^n$ be power series with radius of convergence $\ge r > 0$. Put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \,;$$

then prove that both power series

$$\sum (a_n + b_n)(z-a)^n$$

and

$$\sum c_n (z-a)^n$$

have radius of convergence $\ge r$, and for $|z-a| < r$ the following hold :

$$\sum (a_n + b_n)(z-a)^n = \left[\sum a_n (z-a)^n + \sum b_n (z-a)^n\right]$$

$$\sum c_n (z-a)^n = \left[\sum a_n (z-a)^n\right]\left[\sum b_n (z-a)^n\right].$$

10. Let G and Ω be open subsets of $\mathbb{C}$. Suppose that $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are continuous functions such that $f(G) \subset \Omega$ and $g(f(z)) = z$ for all z in G. If g is differentiable and $g'(z) \neq 0$, then prove that f is differentiable and

$$f'(z) = \frac{1}{g'(f(z))}.$$

11. If a Möbius transformation T takes a circle Γ_1 onto the circle Γ_2 then prove that any pair of points symmetric with respect to Γ_1 are mapped by T onto a pair of points symmetric with respect to Γ_2.

MODULE II

12. State and prove the Fundamental Theorem of Algebra.

13. If G is a region and $f : G \to \mathbb{C}$ is an analytic function such that there is a point $a \in G$ with

$$|f(a)| \geq |f(z)| \text{ for all } z \in G,$$

then prove that f is constant.

14. Let G be simply connected and let $f : G \to \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any z in G. Then prove that there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = \exp g(z)$.

MODULE III

15. If f has an isolated singularity at a then prove that $z = a$ is a removable singularity if and only if $\lim_{x \to a} (z-a) f(z) = 0$.

16. Let $z = a$ be an isolated singularity of f and let

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$$

be its Laurent Expansion in ann $(a\,; 0, R)$. Then prove that $z = a$ is a removable singularity if and only if $a_n = 0$ for $n \leq -1$.

Turn over

17. Show that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx = \frac{\pi}{\sqrt{2}}.$$

(6 × 2 = 12 weightage)

Part C (Essay Type Questions)

Answer any **two** *questions.*
Each question carries a weightage 5.

18. For a given power series

$$\sum_{n=0}^{\infty} a_n \left(z-a\right)^n$$

define the number $\mathrm{R}, 0 \le \mathrm{R} \le \infty$, by

$$\frac{1}{\mathrm{R}} = \limsup \left| a_n \right|^{1/n}.$$

Then prove that

(a) if $\left| z-a \right| < \mathrm{R}$, the series converges absolutely;

(b) if $\left| z-a \right| > \mathrm{R}$, the terms of the series become unbounded and so the series diverges;

(c) if $0 < r < \mathrm{R}$ then the series converges uniformly on $\left\{z : \left| z \right| \le r\right\}$.

19. Let G be an open subset of the plane and $f : \mathrm{G} \to \mathrm{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n\left(\gamma\,; w\right) = 0$ for all $w \in \mathrm{C} \sim \mathrm{G}$, then prove that for $a \in \mathrm{G} \sim \{\gamma\}$,

$$n\left(\gamma\,; a\right) f\left(a\right) = \frac{1}{2\pi i} \int_{\gamma} \frac{f\left(z\right)}{z-a}\, dz.$$

20. State and prove Goursat's Theorem.

21. (a) State and prove Casorati-Weierstrass theorem.

(b) Let f be analytic in the region G except for the isolated singularities $a_1, a_2, \ldots, a_m$. If γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if $\gamma \approx 0$ in G. Then prove that

$$\frac{1}{2\pi i} \int_{\gamma} f = \sum_{k=1}^{m} n(\gamma ; a_k) \operatorname{Res}(f, a_k).$$

(2 × 5 = 10 weightage)