

D 132040

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Name.....

Reg. No.....

**THIRD SEMESTER M.Sc. DEGREE [REGULAR/SUPPLEMENTARY]
EXAMINATION, NOVEMBER 2025**

(CBCSS)

Mathematics

MTH3C13—FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours

Maximum : 30 Weightage

Part A*Answer all questions.**Each question has weightage 1.*

1. Define norm on a linear space. Give an example of a norm on l_1 .
2. If O is an open set, then prove that the set $F = O^c$ is closed.
3. State and prove parallelogram law in the case of an inner product space.
4. Prove that for any orthonormal system $\{e_i\}_{i \geq 1} \subset H$, and for every $x \in H$,

$$\sum_{i \geq 1} |\langle x, e_i \rangle|^2 \leq \|x\|^2.$$

5. Let H be an inner product space. Describe all pairs of vectors x, y for which

$$\|x + y\| = \|x\| + \|y\|.$$

6. For non-zero linear functionals f, g and $\ker f = \ker g$, show that there exists $\lambda \neq 0$ such that $\lambda f = g$.

Turn over

7. Let L be a closed subspace. Consider the subspaces $L^\perp \longleftrightarrow X^*$ and

$$(L^\perp)_\perp = \{x \in X : f(x) = 0, \text{ for all } f \in L^\perp\}.$$

Then, prove that

$$(L^\perp)_\perp = L.$$

8. State and prove the Banach open map theorem.

(8 × 1 = 8 weightage)

Part B

Answer **six** questions choosing two from each unit.

Each question has weightage 2.

UNIT I

9. Let $1 < p < \infty$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all functions f, g on an interval $[a, b]$ such

that the integrals $\int_a^b |f(t)|^p dt$, $\int_a^b |g(t)|^q dt$ and $\int_a^b |f(t) g(t)| dt$ exist prove that

$$\int_a^b |f(t) g(t)| dt \leq \left(\int_a^b |f(t)|^p dt \right)^{1/p} \left(\int_a^b |g(t)|^q dt \right)^{1/q}$$

10. Prove that $C[a, b]$ is complete.
11. Is a quotient space a normed space? Justify your answer.

Unit II

12. Prove that the system $\left\{ \frac{1}{\sqrt{2\pi}} e^{int} \right\}_{n=-\infty}^{\infty}$ is an orthonormal basis of $L_2[-\pi, \pi]$.

13. If E is a closed subspace of H and $\text{codim } E = 1$, then prove that the subspace $E^\perp$ is 1-dimensional.
14. Prove that f is a bounded functional if and only if f is a continuous functional.

Unit III

15. Prove that for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $(l_p)^* = l_{p^*}$.
16. Prove the set $K(X \mapsto Y)$ of compact operators from X to Y is a linear sub-space of $L(X \mapsto Y)$.
17. If $\|A\| < 1$, then prove that $(I - A)$ is invertible and $(I - A)^{-1} = \sum_0^\infty A^k$.

(6 × 2 = 12 weightage)

Part C

Answer **two** questions.

Each question has weightage 5.

18. Let E, X_i for $i = 1, 2$ be linear normed linear spaces with X_i being complete and $T_i : E \mapsto X_i$ isometries into dense subspaces of X_i for $i = 1, 2$. Then prove that the natural mapping $T_2 \circ T_1^{-1} : T_1 E \mapsto T_2 E$ can be extended to an isometry between X_1 and X_2 .
19. State and prove necessary condition for a Hilbert space to have an orthonormal basis.
20. Let E be an n -dimensional normed space. Prove that E is complete.
21. Prove that M is relatively compact if and only if for every $\varepsilon > 0$ there exists a finite ε -net in M .

(2 × 5 = 10 weightage)