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D 143201 **(Pages : 4)** **Name..................................**

Reg. No...............................

FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2026

(CBCSS)

Mathematics

MTH4C15—ADVANCED FUNCTIONAL ANALYSIS

(2019 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Define spectrum of a bounded operator on a complex Banach Space.
2. Prove that for every compact operator T, $O \varepsilon \sigma(T)$.
3. Prove that the point spectrum σ_p of a self operator A on a Hilbert space satisfies $\sigma_p(A) \subseteq R$, where R is the field of real numbers.
4. Show that if P is an orthogonal projection on a Hilbert space H, then $I_m P \perp \ker P$.
5. With usual notations, prove that E_λ are orthogonal projections.
6. State Hörmander theorem.
7. Give an example of a set which is convex but is not perfectly convex.
8. Prove that if the elements x and xy in an algebra are invertible, then the element y is invertible.

(8 × 1 = 8 weightage)

Turn over

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Part B

Answer six questions by choosing any **two** *questions from each unit.*

Each question carries 2 weightage.

Unit I

9. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigen values of a bounded operator A on a complex Banach space X. If $Ax_i = \lambda_i x_i$ for $i = 1, 2, \ldots, n$ and $x_i \neq 0$, then $x_1, x_2, \ldots x_n$ are linearly independent.

10. Let X be a complex Banach space and let $A \varepsilon L(X)$ for $\lambda \varepsilon C$, assume that there exists a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $\|x_n\| = 1$ and $Ax_n - \lambda x_n \to 0$ as $n \to \infty$. Prove that $\lambda \varepsilon \sigma(A)$.

11. Let H be a Hilbert space and $A : H \to H$ be a bounded operator on H. Prove that $\langle Ax, x \rangle \varepsilon R$ for any $x \varepsilon H$ if and only if A is symmetric.

Unit II

12. Let A be a bounded operator on a Hilbert space H. Show that if $A \geq 0$, then for any polynomial $p(\lambda)$ with non-negative co-efficients we have $p(A) \geq 0$.

13. Let P_1 and P_2 be two orthogonal projections and let $P_1 P_2 = P_2 P_1 = P$. Show that P is an orthogonal projection with $E = \text{Im } P = E_1 \cap E_2$ where $E_i = \text{Im } P_i ; i = 1, 2$.

14. State Hilbert theorem on the spectral decomposition of self-adjoint bounded operators on a Hilbert space.

Unit III

15. Let X be a Banach space over the field of complex numbers. Let $A \subset X^*$ be a set of bounded linear functionals such that for every $x \varepsilon X$ the set $\{f(x) : f \varepsilon A\}$ is bounded. Show that A is a bounded set in X*.

16. Let T_1, T_2 be perfectly convex sets in a Banach space X. Let $u \,\varepsilon\, X$ and assume that T_2 is bounded. Prove that the sets $T_1 + T_2$ and $T_1 + u$ are perfectly convex.

17. Let $\mathcal{A}$ be a Banach algebra. Prove that for every proper ideal $I \subseteq \mathcal{A}$ there exists a maximal ideal $\mathcal{M}$ such that $I \subseteq \mathcal{M}$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries 5 *weightage.*

18. (a) Let X be an infinite dimensional Banach space and $T : X \to X$ be a compact operator. For $\lambda \neq 0$, let T_λ denote the operator $T - \lambda I$ and $\Delta_\lambda = \operatorname{Im} T_\lambda$. Prove that $\Delta_\lambda = \bar{\Delta}_\lambda$.

    (b) Let X and Y be Banach spaces. Let $A : X \to Y$ be a closed graph operator and Dom A = X. Show that A is continuous.

19. (a) Let $A, A_0, A_1, A_2, \ldots, A_n$ be bounded operators on a Hilbert space H with $A_0 \leq A_1 \leq \ldots \leq A_n \leq A$. Show that there exists a bounded operator B on H and $A_n{}^x \to Bx$ for all $x \,\varepsilon\, H$.

    (b) Let the operator S_r be the right shift on l_2 defined by $S_r\,(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Find the spectrum of S_r.

20. (a) State and Prove Hahn-Banach theorem for complex valued linear functionals.

    (b) Let the operator K : $L_2\,[0,1] \to L_2\,[0,1]$ be given by $(kf)(t) = \int_0^1 k(t,s)\, f(s)\, ds$, where $k(t,s) = \max\{t, s\}$ for $0 \leq t, s \leq 1$. Prove that k is a compact self-adjoint operator.

Turn over

21. (a) Let X and Y be Banach spaces. Show that a sequence of operators $T_n \,\varepsilon\, L(X, Y)$ converges strongly to an operator $T \,\varepsilon\, L(X, Y)$ if and only if :

(i) The sequence $\{T_n{}^x\}$ converges for any x from a dense subset of X.

(ii) There exists $c > 0$ such that $\|T_n\| \le c$.

(b) Let $\mathcal{A}$ be a Banach algebra. Show that if $x \,\varepsilon\, \mathcal{A}$ with $\|x\| < 1$, then $e - x$ is invertible and $(e - x)^{-1} = \sum_{k=0}^{\infty} x^k$.

(2 × 5 = 10 weightage)