

D 143206

(Pages : 3)

Name.....

Reg. No.....

**FOURTH SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY)
EXAMINATION, APRIL 2026**

(CBCSS)

Mathematics

MTH4E08—COMMUTATIVE ALGEBRA

(2019 Admission onwards)

Time : Three Hours

Maximum : 30 Weightage

Part A*Answer all questions.**Each question has weightage 1.*

1. Let A be a ring and $\mathfrak{a}$ be an ideal of A . Verify whether $f : A \rightarrow A/\mathfrak{a}$ defined by $f(x) = x + \mathfrak{a}$ is a ring homomorphism.
2. Verify whether the ideal generated by 10 is maximal in the ring $\mathbb{Z}$ of integers.
3. Let $\mathfrak{a} = (3)$ be the ideal generated by 3 in the ring $\mathbb{Z}$ of integers. Find an ideal $\mathfrak{b}$ such that $\mathfrak{a} + \mathfrak{b} = (1)$.
4. Let $\mathfrak{p}$ be a prime ideal of a ring A . Verify whether $S = A - \mathfrak{p}$ is a multiplicatively closed subset of A .
5. Let $A = \mathbb{Z}$ and $S = A - \mathfrak{p}$ where $\mathfrak{p}$ is the ideal generated by 5. Verify whether $2/3 \in S^{-1}A$.
6. Let M be an A -module and S be a multiplicatively closed subset of A . Describe the module $S^{-1}M$.
7. Let $B = k[x, y]$ and $A = k[x]$ be polynomial rings over a field k . Verify whether y is integral over A .
8. Verify whether the polynomial ring $k[x]$ where k is a field satisfies a.c.c. on ideals.

(8 × 1 = 8 weightage)

Turn over

Part B

*Answer any **two** questions from each module.*

Each question has weightage 2.

MODULE I

9. Let $f : A \rightarrow B$ be a ring homomorphism from A onto B . Show that $A/\ker f$ is isomorphic to B .
10. Let m be an ideal of a ring A such that for all $x \in A - m$, x is a unit. Show that m is a maximal ideal of A .
11. Let $\mathcal{R}$ be the Jacobson radical of a ring A . Show that if $x \in \mathcal{R}$ then $1 - xy$ is a unit in A for every $y \in A$.

MODULE II

12. Let S be a multiplicatively closed subset of a ring A and let $f : A \rightarrow S^{-1}A$ be given by $f(a) = a/1$. Show that every element of $S^{-1}A$ is of the form $f(a)(f(s))^{-1}$ for some $a \in A$ and $s \in S$.
13. Let N, P be submodules of an A -module M . Show that $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.
14. Let $\phi : M \rightarrow N$ be an A -module homomorphism. Show that if (ϕ) is injective then for all prime ideals p of A the homomorphism $\phi_p : M_p \rightarrow N_p$ is also injective.

MODULE III

15. Let A be a subring of a ring B . Show that

$$C = \{x \in B : x \text{ is integral over } A\}$$

is a subring of B .

16. Let $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \rightarrow 0$ be an exact sequence of A -modules. Show that if M is Noetherian then M' and M'' are Noetherian.
17. Let S be a multiplicative subset of a ring A . Show that if A is Noetherian then $S^{-1}A$ is Noetherian.

(6 × 2 = 12 weightage)

Part C

Answer any **two** questions.

Each question has weightage 5.

18. (a) Let $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ be an exact sequence of A -modules. Show that

- (i) f is injective,
- (ii) g is surjective.

(b) Show that a sequence of A -modules $M' \xrightarrow{u} M \xrightarrow{v} M'' \rightarrow 0$ is exact if and only if for every A -module N the sequence

$$0 \rightarrow \text{Hom}(M'', N) \xrightarrow{v^*} \text{Hom}(M, N) \xrightarrow{u^*} \text{Hom}(M', N)$$

is exact.

19. (a) Define primary ideal of a ring A .

(b) Let q be a primary ideal of a ring A . Show that $r(q)$ is the smallest prime ideal containing q .

(c) Let $A = k[x, y]$ where k is a field and q be the ideal generated by $\{x, y^2\}$. Show that q is a primary ideal of A .

20. (a) Let $a = \cap q_i$ be a minimal primary decomposition of an ideal a . Let $p_i = r(q_i)$ for each i . Show that for each i , $p_i = r(a : x)$ for some $x \in A$.

(b) Let S be a multiplicatively closed subset of a ring A and q be a p -primary ideal of A . Show that if $S \cap p \neq \emptyset$ then $S^{-1}q = S^{-1}A$.

21. Let M be an A -module. Show that

- (a) M is Noetherian if and only if every submodule of M is finitely generated.
- (b) If M satisfies both a.c.c and d.c.c. on submodules then M has a composition series.

(2 × 5 = 10 weightage)